Dissipative quadratizations of polynomial ODE systems - demo presentation: DQbee

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Quadratization: what is it?

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DONE!

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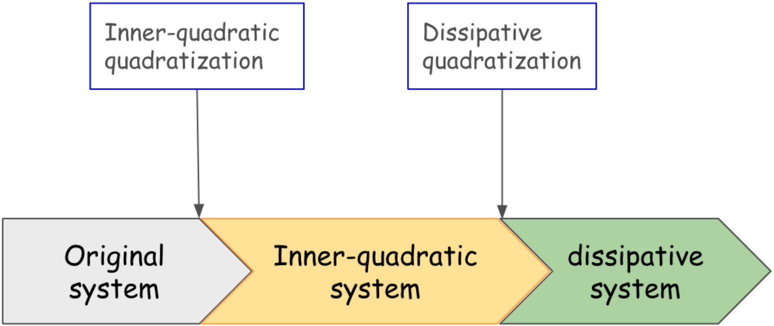
Important fact: Dissipativity at equilibrium $\mathbf{x}^* \implies$ Asymptotic stability at $\mathbf{x}^*$
(i.e. exponential convergence to $\mathbf{x}^*$ in a small neighbourhood)

Dissipative quadratization?

What: $\boxed{\text{Quadratization that perserve dissipativity.}}$



DQbee: a Python package computes dissipative quadratization:

- Fast.
- Easy to use.
- Great visualization.
- Optimal.

Code link: https://github.com/yubocai-poly/DQbee

How far it goes: Coupled duffing oscillators

Consider a coupled duffing system with n oscillators where $A \in \mathbb{R}^{n\times n}$, $\delta \in \mathbb{R}$:

$$\mathbf{x}'' = A\mathbf{x} - (A\mathbf{x})^3 - \delta\mathbf{x}',$$

Transfer to first order: $\mathbf{z} = [z_1, \dots, z_n]^\top$ for the derivatives of $\mathbf{x}$:

$$\dot{\mathbf{x}} = \mathbf{z}, \quad \mathbf{z}' = A\mathbf{x} - (A\mathbf{x})^3 - \delta\mathbf{z}.$$

We have 2^n dissipative equilibria.

Setting: $n = 1, \dots, 8$ taking $\delta = 2$ and A being the tridiagonal matrix with ones on the diagonal and $\frac{1}{3}$ on the adjacent diagonals.

How far it goes: results

n	dimension	# equilibria	# new vars	time (inner-quadratic)	time (dissipative)	
					NUMPY	ROUTH-HURWITZ
1	2	2	1	0.02	0.05	0.07
2	4	4	2	0.07	0.19	0.65
3	6	8	4	0.20	0.74	36.57
4	8	16	5	0.39	1.62	1179.33
5	10	32	7	0.72	4.30	> 2000
6	12	64	9	1.20	11.28	> 2000
7	14	128	10	1.75	28.23	> 2000
8	16	256	12	2.63	78.70	> 2000

Table: Runtimes (in seconds) for n coupled Duffing oscillators, results were obtained on a laptop with the following parameters: Apple M2 Pro CPU @ 3.2 GHz, MacOS Ventura 13.3.1, CPython 3.9.1.

Thank you for your attention!

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Figure: Paper

Figure: Code

Today 4:30 pm - Room: Hollenfels - TACAS: Simulation