Dissipative quadratizations of polynomial ODE systems

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Quadratization: What?

Consider a **polynomial** system of differential equations

$$\mathbf{x}' = \mathbf{p}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = \mathbf{x}(t) = (x_1(t), \dots, x_n(t))$ is a vector of unknown functions and $\mathbf{p} = (p_1, \dots, p_n)$ s.t. $p_1, \dots, p_n \in \mathbb{R}[\mathbf{x}]$.

New variables $y_1 = g_1(\mathbf{x}), \cdots, y_m = g_m(\mathbf{x})$ are called **quadratization** if there exist

$$\begin{cases} \mathbf{q}_1(\mathbf{x}, \mathbf{y}) = (q_{1,1}(\mathbf{x}), \dots, q_{1,n}(\mathbf{y})) \\ \mathbf{q}_2(\mathbf{x}, \mathbf{y}) = (q_{2,1}(\mathbf{x}, \mathbf{y}), \dots, q_{2,m}(\mathbf{x}, \mathbf{y})) \end{cases}$$

such that $\deg \mathbf{q}_1, \deg \mathbf{q}_2 \leqslant 2$, we have

$$\mathbf{x}' = \mathbf{q}_1(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad \mathbf{y}' = \mathbf{q}_2(\mathbf{x}, \mathbf{y})$$

Toy example

$$x' = x^4 \;(\text{degree} = 4) \quad \xrightarrow{\text{introduce } y := x^3} \quad \begin{cases} x' = xy \\ y' = 3x'x^2 = 3x^6 = \underline{3y^2} \end{cases} \;(\text{degree} \leqslant 2)$$

Quadratization: Why?

- Synthesis of chemical reaction networks:

  deg $\leqslant 2 \iff$ bimolecular network
- Reachability analysis: explicit error bounds for Carleman lineariza -tion in the quadratic case.
- Moder Order Reduction (MOR)

Research objectives

Find the quadratization that preserves the **numerical** properties of the original system.

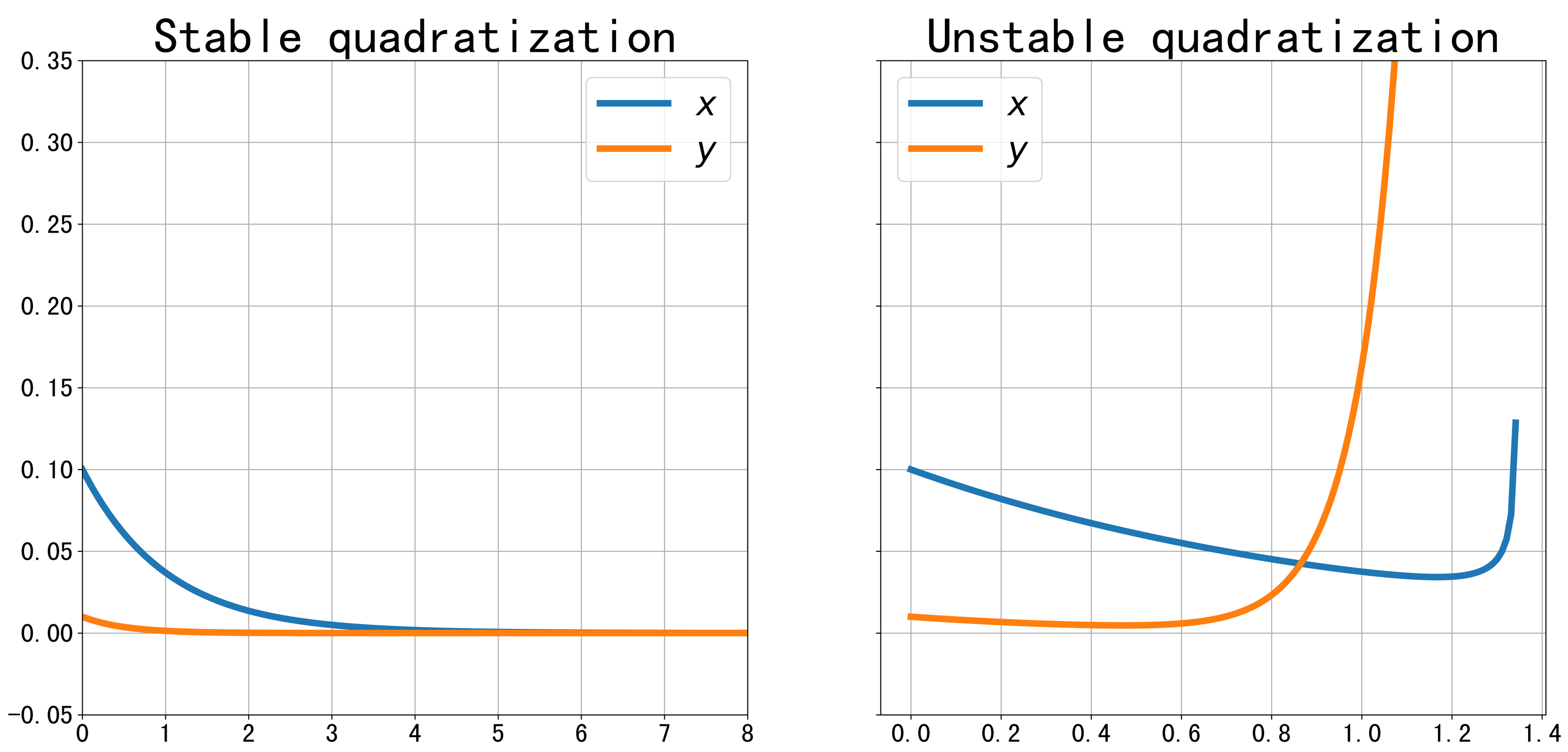


Figure 1. Initial condition: $\mathcal{X}_0 = [x_0, y_0 = x_0^2] = [0.1, 0.01]$.

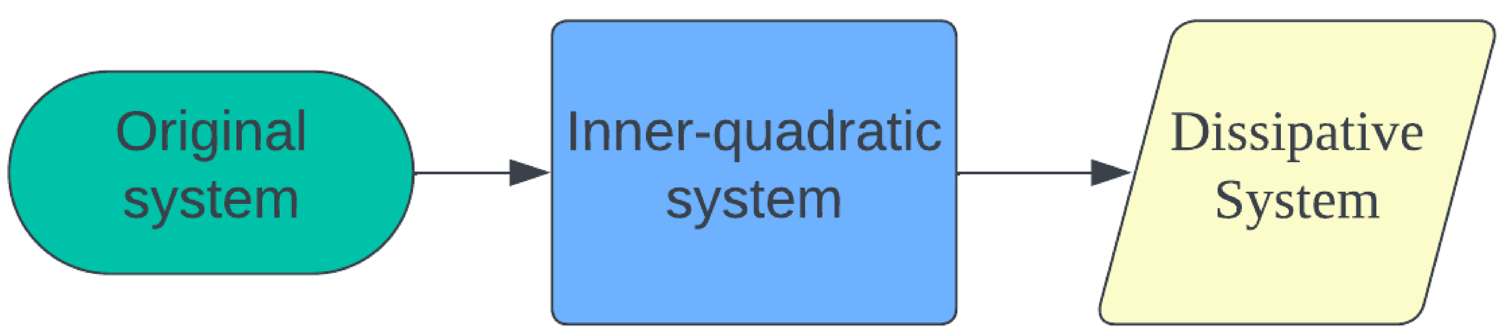
Stable:

$$\begin{cases} x' = -x + xy \\ y' = -2y + 2y^2 \end{cases}$$

Unstable:

$$\begin{cases} x' = -x + xy \\ y' = -2y + 2y^2 + 12(y - x^2) \end{cases}$$

Our Methodology



Consider an equation:

$$x' = -x(x-1)(x-2)$$

- Equilibria: 0, 1, 2
- Dissipative Equilibria: $x = 0$ and $x = 2$

Inner-quadratic quadratization: introduce $y = x^2$:

$$\begin{cases} x' = -xy + 3x^2 - 2x \\ y' = -2y^2 + 6xy - 4x^2 \end{cases}$$

Stabilizer: $y - x^2$

Dissipative quadratization:

$$\begin{cases} x' = -xy + 3x^2 - 2x \\ y' = -2y^2 + 6xy - 4x^2 - \lambda(y - x^2) \end{cases}$$

Jacobian:

$$J = \begin{bmatrix} -y + 6x - 2 & -x \\ 6y - 8x & -4y + 6x \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 \\ -2x & 1 \end{bmatrix}$$

For $\lambda = 1, 2, 4, 8, \cdots$, eigenvalues table:

λ	at $(0,0)$	at $(2,4)$
1	$-2, -1$	$-2, \mathbf{3}$
2	$-2, -2$	$-2, \mathbf{2}$
4	$-2, -4$	$-2, \mathbf{0}$
8	$-2, -8$	$\mathbf{-2, -4}$

Applications

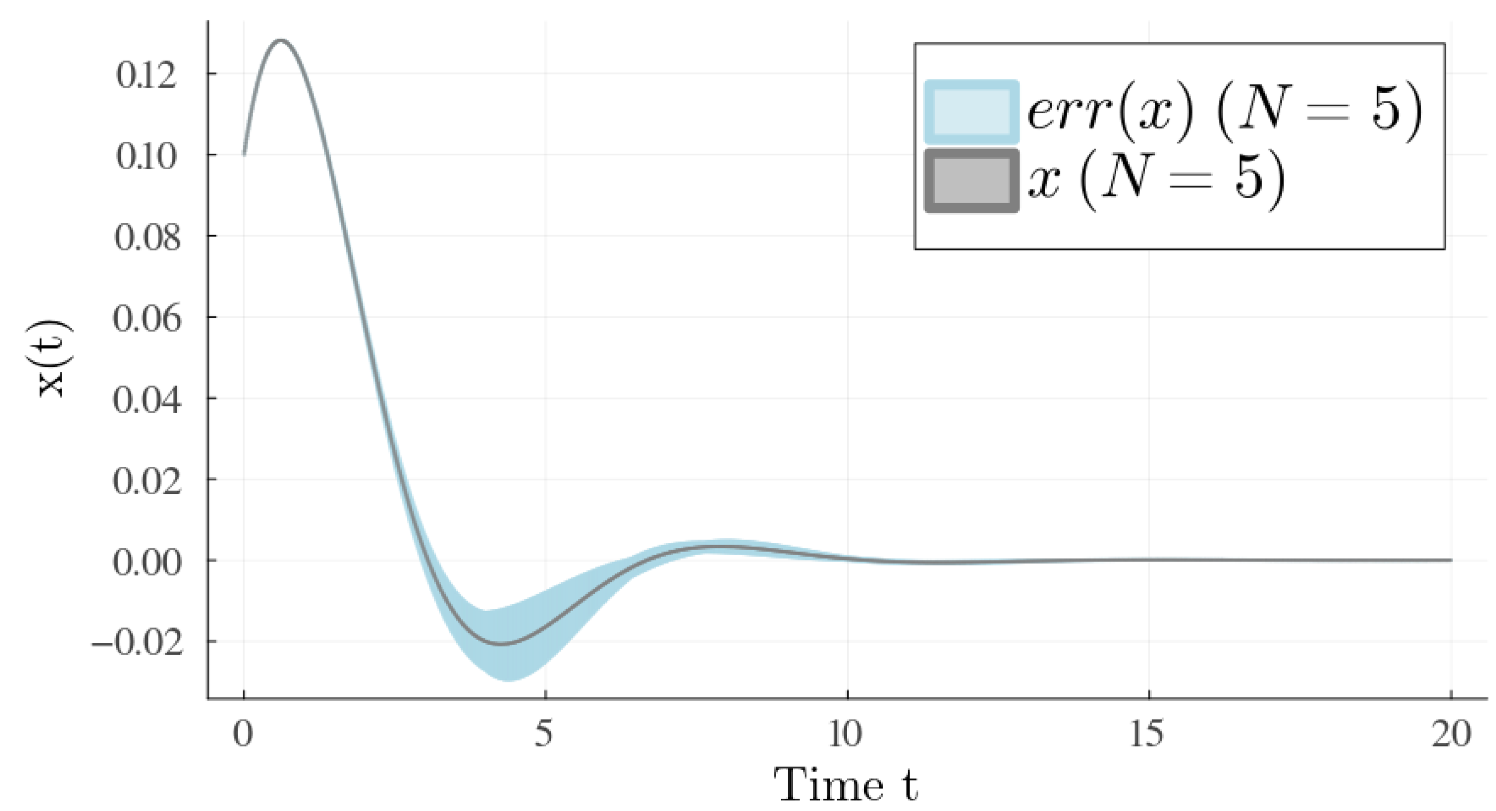
- Preserving bistability
- Reachability analysis with Carleman linearization:

Duffing equation $\Rightarrow x'' = x + x^3 - x'$

First order $\Rightarrow x_1' = x_2, \quad x_2' = x_1 + x_1^3 - x_2$

Dissipative quadratization $\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_1 y + x_1 - x_2 \\ y' = -y + x_1^2 + 2x_1 x_2 \end{cases}$

Initial conditions $x_1(0) = 0.1, x_2(0) = 0.1, y(0) = x_1(0)^2 = 0.01$ and truncation order $N = 5$.



More information

- Paper: https://arxiv.org/abs/2311.02508
- Code: https://github.com/yubocai-poly/DQbee

Figure 2. Paper

Figure 3. Code